# Laplace equation - Numerical example

The Laplace equation is given as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

The scalar function $\phi$ can be f.ex. velocity potential or temperature. We will look at both.

# Laplace equation  - Numerical example

With temperature as input, the equation describes two-dimensional, steady heat conduction.

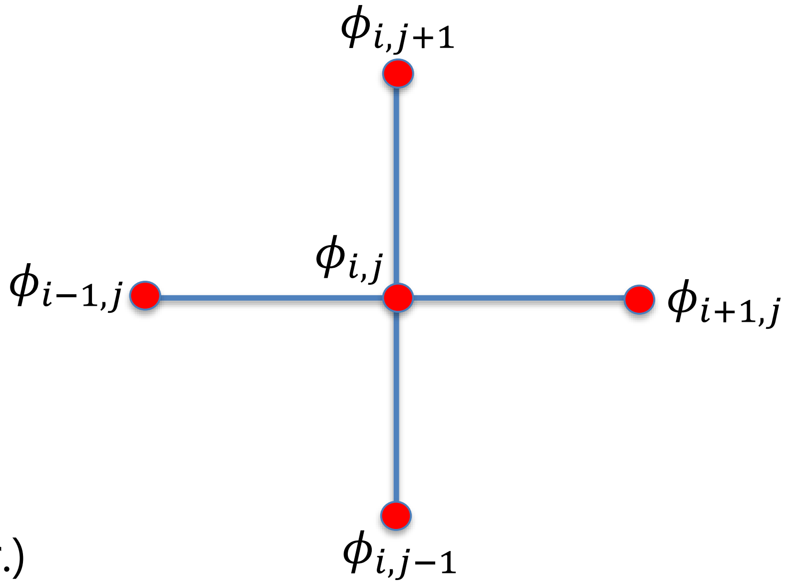
The velocity and its potential is related as $u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$,
where u and v are velocity components in x- and y-direction respectively. For flow, it requires *incompressible*, *irrotational*, *non-viscous* and *steady* conditions

As will be seen, it is easiest to start with temperature.

# Laplace equation - Numerical example

The interior domain can be split up as shown in the figure below:

Numerical molecule:




Discretization:

$$\frac{\partial \phi}{\partial x} \approx \frac{1}{\Delta x}\left(\phi_{i+1,j} - \phi_{i,j}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{1}{\Delta x^2}\left(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}\right)$$ (same for y-dir.)

# Laplace equation - Numerical example

Adding the discretized differentials together gives us:

$$2 \cdot \left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right] \phi_{i,j} = \phi_{i-1,j} + \phi_{i,j-1} + \left(\frac{\Delta x}{\Delta y}\right)^2 \left(\phi_{i+1,j} + \phi_{i,j+1}\right)$$

Show this yourself !

So we have an explicit expression for point *i,j* that contains its neighbors.

For simplicity we continue with uniform grid -> $\Delta x = \Delta y$

# Laplace equation - Numerical example

The algebraic equation system we must solve gets:

$$-4 \cdot \phi_{i,j} + \phi_{i-1,j} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i,j+1} = 0$$

Consider temperature first !

# Laplace equation - *Physical domain*

$$\frac{\partial T}{\partial y} = 0 \rightarrow T_{j=N} = T_{j=N-1}$$

$T = 200K$

$T = 300K$

$$\frac{\partial T}{\partial y} = 0 \rightarrow T_{j=1} = T_{j=2}$$

# Laplace equation - *Boundary conditions*

Easiest to start with is temperature, because the directly solved variable from the scalar equation is what we are interested in. Two different BCs:

Dirichlet: $T_{bc}$ is given. Constant temperature at any boundary.

Neumann: $\left(\frac{\partial T}{\partial x_i}\right)_n$ The normal gradient is given. Heat flux.

# Laplace equation - Numerical example

$$T_{1,1} = 200$$

$$T_{2,5} = T_{2,4}$$

$$\ldots$$

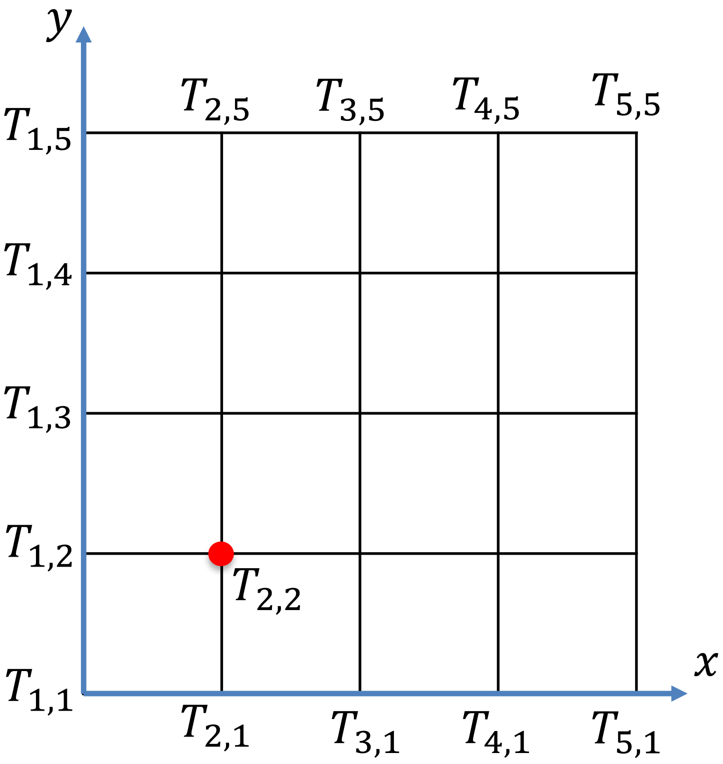
$$T_{2,2} = \frac{1}{4}\left[T_{1,2} + T_{2,1} + T_{3,2} + T_{2,3}\right]$$

$$\ldots$$

$$T_{4,5} = T_{4,4}$$

$$\ldots$$

$$T_{5,1} = 300$$



# Laplace equation - *Solution (Gauss-Seidel SOR)*

$$T_{i,j} = \frac{1}{4}\left[T_{i-1,j} + T_{i,j-1} + T_{i+1,j} + T_{i,j+1}\right]$$

$$T_{i,j}^{new} = T_{i,j}^{old} + \alpha \cdot \underbrace{\left[\frac{1}{4}\left(T_{i-1,j} + T_{i,j-1} + T_{i+1,j} + T_{i,j+1}\right) - T_{i,j}^{old}\right]}_{\delta T}$$

We added and subtracted the old value of $T_{i,j}$ on the right hand side. Inside brackets is the change in $T_{i,j}$ for the current iteration. $\alpha$ is the over-relaxation parameter.

# Laplace equation  - Numerical example

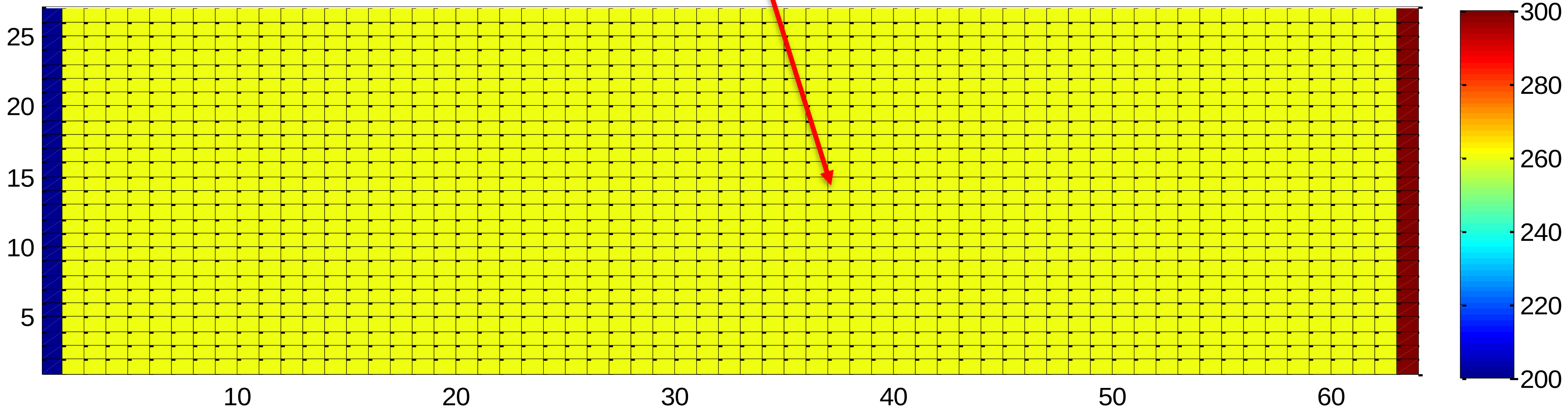
Remember to either exclude the indexes for boundary values, or overwrite/enforce them inside the iteration loop.

Check the difference between old and new values (residual). Several ways to calculate this !

$x$

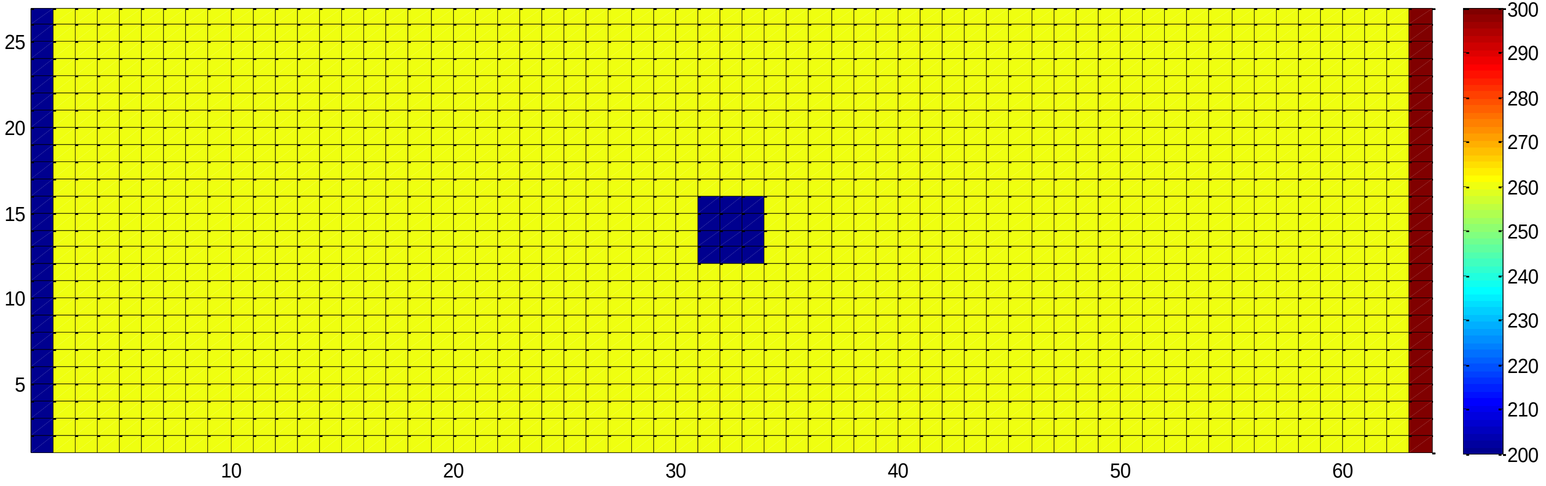
# Laplace equation - *Numerical domain*

$Start\ values\!: T_{start} = 260\ K$

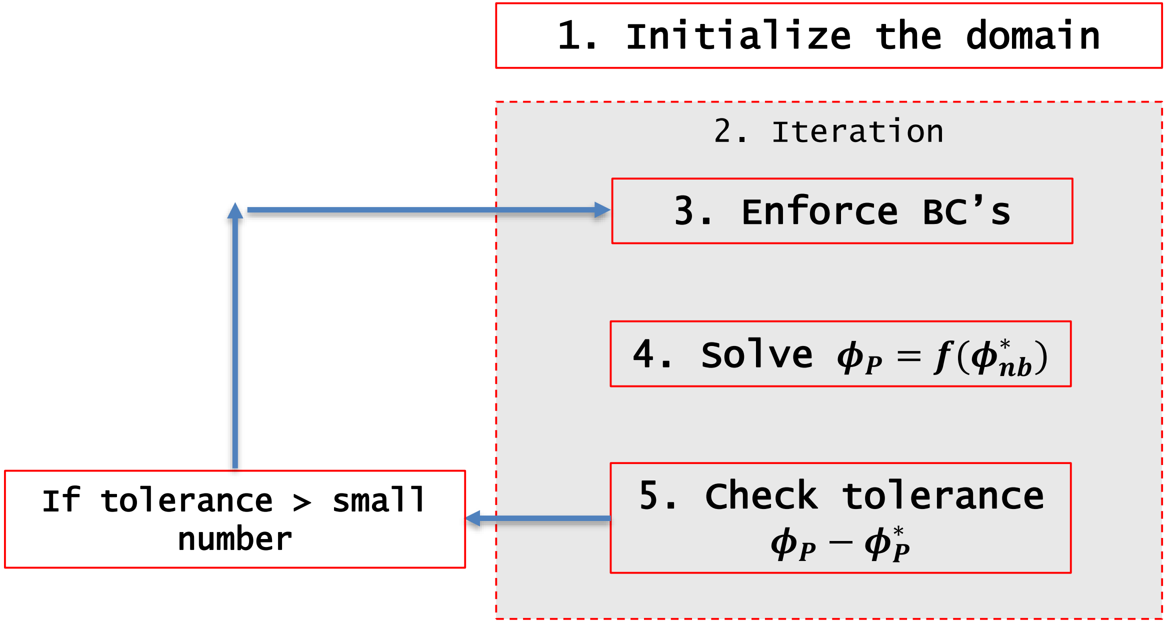


# Laplace equation - *Numerical domain*

Box inside, T=200K



# Laplace equation - *Solution procedure*

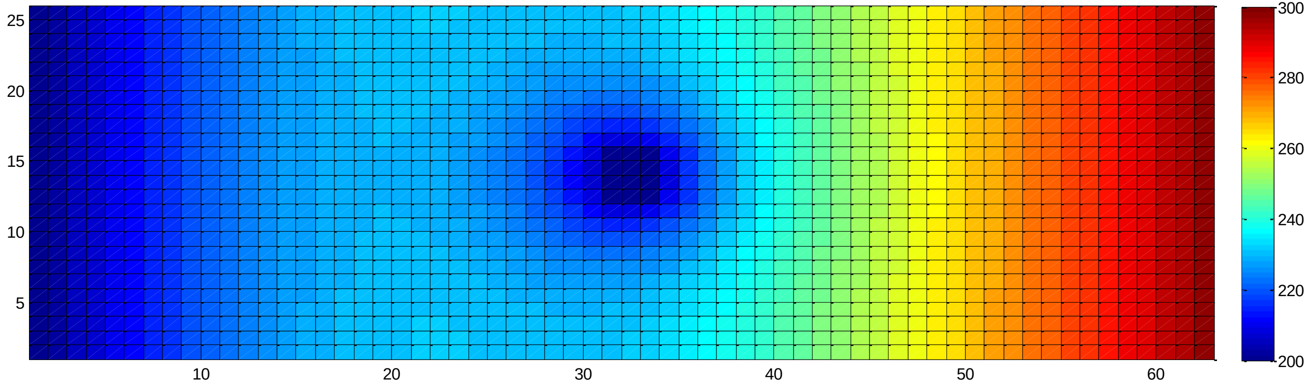



Convergence:

$$\frac{\max\left(\left|\delta T_{i,j}\right|\right)}{\max\left(\left|T_{i,j}\right|\right)} < \epsilon$$

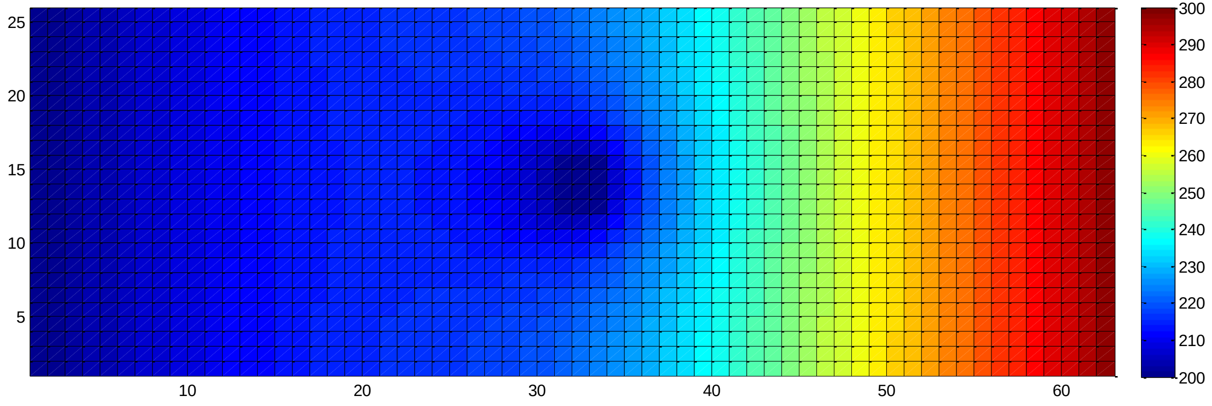
Where $\epsilon$ is a small number

# Laplace equation - *Results*




500 iterations

1000 iterations



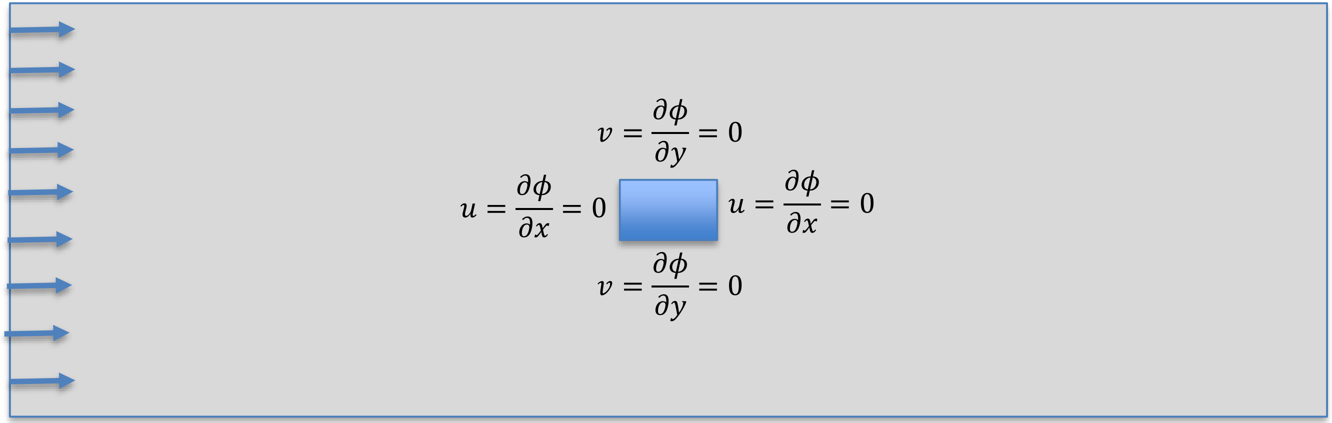
# Laplace equation - Velocity potential

We wish to study external flow. That is flow around an object. The easiest is a rectangular box. The domain will be like below:

$$v = \frac{\partial \phi}{\partial y} = 0$$

$u = U_{in} \rightarrow$

$$\frac{\partial \phi}{\partial x} = U_{in}$$




$$\frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$v = \frac{\partial \phi}{\partial y} = 0$$

# Laplace equation - Velocity potential

It is solved exactly the same way as with temperature, but the boundary conditions are different. Also, *Dirichlet* conditions for *u* and *v* gives *Neumann* conditions for the velocity potential.

Inlet: $\phi_{1,j} = \phi_{2,j} - U_{in} \cdot \Delta x$

Outlet: $\phi_{N,j} = 2 \cdot \phi_{N-1,j} - \phi_{N-2,j}$

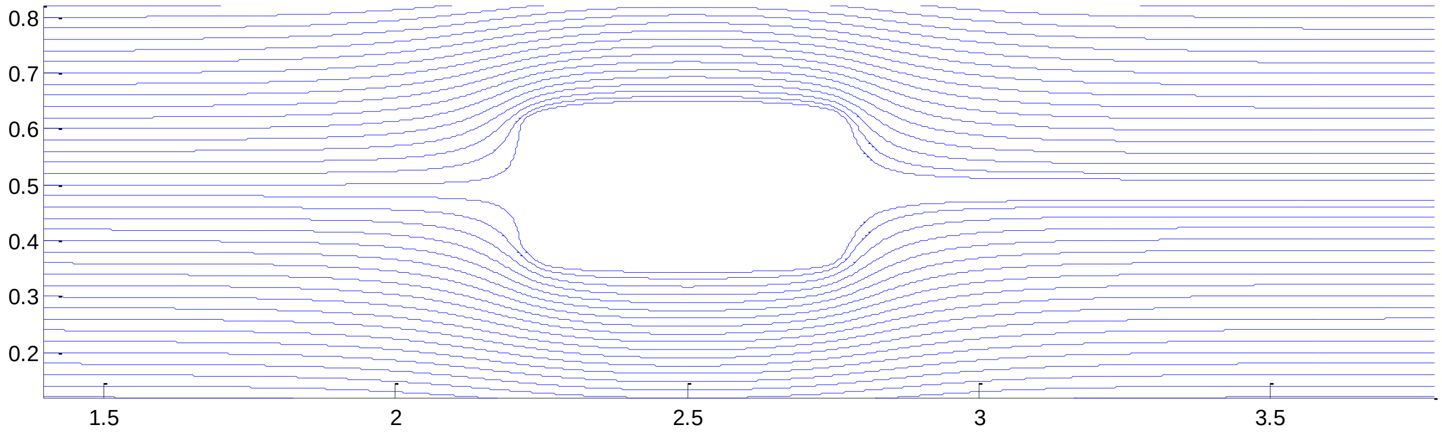
Top: $\phi_w = \phi_{w-1}$ Bottom: $\phi_w = \phi_{w+1}$

At the top and bottom use v=0

Correct initialization is important here ($U_{in}$ all over the domain)

# Laplace equation - *Results*

Streamlines




u-velocity